View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

J. Math. Anal. Appl. 362 (2010) 415–426

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Analysis of a new variational model for multiplicative noise removal ☆

Zhengmeng Jin [a,b,*], Xiaoping Yang [a]

[a] School of Science, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, PR China
[b] School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, Jiangsu, PR China

ARTICLE INFO

Article history:
Received 10 December 2008
Available online 22 August 2009
Submitted by P. Sacks

Keywords:
Calculus of variation
Weak solutions
BV
Variational model
Multiplicative noise
Image restoration

ABSTRACT

In this paper we consider a new variational model for multiplicative noise removal. We prove the existence and uniqueness of the minimizer for the variational problem. Furthermore, we derive the existence and uniqueness of weak solutions for the associated evolution equation. Finally, some numerical experiments are shown to compare the proposed model with the model given by Aubert and Aujol.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

As is well known, multiplicative noises are commonly found in many real world image processing applications, such as in laser images, microscope images, medical ultrasonic images and SAR images. Unlike additive noises, these noises are much more difficult to be removed from the corrupted image, mainly not only because of their multiplicative nature, but also because of their distributions which are generally not Gaussian. The additive noise models have been extensively studied over the last decades, such as the PDE-based variational methods including the ROF model [12] and LLT model [10]. However, the multiplicative noises have been studied very little. Let f be an observed image with multiplicative noise defined on Ω, where Ω is a rectangle of $\mathbb{R}^2$. The multiplicative model is given by

$$f = un,$$

where u denotes the image to be recovered and n is the noise.

In the literature, there exist two main variational approaches to process multiplicative noise problems. One is proposed by Rudin, Lions and Osher [11], and the other is given by Aubert and Aujol [1]. In [11], under the assumption that the mean of the multiplicative noise is equal to 1 and the variance is known, the authors introduced the following denoising model:

$$\min_u \left\{ J(u) + \lambda_1 \int_\Omega \frac{f}{u} + \lambda_2 \int_\Omega \left(\frac{f}{u} - 1 \right)^2 \right\}, \tag{1.1}$$

☆ This work is supported by the Doctoral Programme Foundation of Education Ministry of China (No. 2003028802) and the Excellent PhD Student Foundation of NUST.

* Corresponding author at: School of Science, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, PR China.
E-mail addresses: jzhm353@yahoo.com.cn (Z. Jin), yangxp@mail.njust.edu.cn (X. Yang).

0022-247X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2009.08.036

where $J(u)=\int_\Omega |Du|$ is the TV regularization term, the last two terms are the data fitting terms, λ_1 and λ_2 are the weighted parameters. They gave some numerical experiment results for the model (1.1). As far as we know, the theoretical analysis about the variational problem (1.1) has not been studied. In [1], based on maximum a posteriori (MAP) regularization approach, Aubert and Aujol derived the denoising model (called AA model) as follows:

$$\min_u\left\{J(u)+\lambda\int_\Omega\left(\log u+\frac{f}{u}\right)\right\}, \tag{1.2}$$

where the last term is the fitting term, λ is the weighted parameter. Notice that AA model (1.2) is specifically devoted to the denoising of images corrupted by Gamma noise. Gamma noise is more complex than Gaussian noise and appear in SAR images [13]. The authors in [1] proved the existence of a minimizer to the variational problem (1.2), and derived existence and uniqueness results of the solution to the associated evolution equation:

$$\partial_t u=\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)+\lambda\frac{f-u}{u^2}. \tag{1.3}$$

Recently, in [6] Huang, Ng and Wen considered an exponential transformation $u\to e^u$ in the fitting term of AA model, and proposed the following denoising model:

$$\min_{u,w}\left\{\int_\Omega\left(u+fe^{-u}\right)+\alpha_1\int_\Omega|u-w|^2+\alpha_2 J(w)\right\}, \tag{1.4}$$

where α_1 and α_2 are positive regularization parameters, and w is an auxiliary variable. They further developed an alternating minimization algorithm for the model (1.4) by incorporating another way of modified TV regularization in [7], and showed the capability of their model on some numerical examples. The theoretical analysis about the variational problem (1.4) was not given in [6].

In this paper, motivated by AA model (1.2) and model (1.4), we study the following denoising model:

$$\min_u\left\{J(u)+\lambda\int_\Omega\left(u+fe^{-u}\right)\right\}. \tag{1.5}$$

We know that the fitting term $\log u+\frac{f}{u}$ of AA model (1.2) is turned into $u+fe^{-u}$ in (1.5) under the exponential transformation: $u\to e^u$. Here the choice of the new fitting term $u+fe^{-u}$ is based on the following two reasons: one is that the exponential transformation preserves image edges well [6]; The other is that $u+fe^{-u}$ is globally convex for all u as $f>0$, which ensures the uniqueness of the solutions to the variational problem (1.5). We investigate the following initial boundary value problem of the evolution equation corresponding to (1.5):

$$\partial_t u=\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)+\lambda\left(fe^{-u}-1\right)\quad\text{on }\Omega_T, \tag{1.6}$$

$$\frac{\partial u}{\partial n}=0\quad\text{on }\partial\Omega\times[0,T], \tag{1.7}$$

$$u(0)=u_0\quad\text{on }\Omega. \tag{1.8}$$

For problem (1.5) we prove the existence and uniqueness of the minimizer in $BV(\Omega)$. For problem (1.6)–(1.8), we show the existence and uniqueness of the weak solution. Our method is to study firstly the approximation problem of (1.6)–(1.8). Then some uniform estimates of the approximation solutions are derived, which enable us to pass to the limit in the approximate problem to get the existence of weak solutions to (1.6)–(1.8). In the last part of this paper some numerical experiments are demonstrated to show the capabilities of the model for multiplicative noise removal.

This paper is organized as follows. In Section 2 we give some classical theory for the space of $BV(\Omega)$ and present the definition of weak solutions to problem (1.6)–(1.8). In Section 3, the existence and uniqueness of the minimizer to problem (1.5) is proved. In Section 4 we study the associated evolution equation and get the existence and uniqueness of weak solutions to problem (1.6)–(1.8). Finally, in Section 5 some numerical experiments are shown to compare the proposed model with the known model – AA model. It is worth mentioning that the interested domain for image processing problem is in $\mathbb{R}^2$. However, our results of this paper hold for a generic domain in $\mathbb{R}^n$.

2. Preliminaries

Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^n$, and write $\Omega_T:=\Omega\times[0,T]$ with $T>0$. In the following we recall some basic notations and facts on the space of $BV(\Omega)$ (see [4,5,9]).

Definition 2.1. Define $BV(\Omega)$ as a space of functions $u \in L^1(\Omega)$ such that the following quantity

$$\int_\Omega |Du| := \sup\left\{ \int_\Omega u \operatorname{div}(\varphi)\, dx \;\middle|\; \varphi \in C_0^1\big(\Omega; \mathbb{R}^n\big),\ |\varphi| \leqslant 1 \right\}$$

is finite. $BV(\Omega)$ is a Banach space with the norm $\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \|u\|_{L^1(\Omega)}$.

About the lower semicontinuity and compactness, we state the following theorems [4].

Theorem 2.1. *Suppose $u_k \in BV(\Omega)$ $(k = 1, \ldots)$ and $u_k \to u$ in $L^1_{loc}(\Omega)$. Then*

$$\int_\Omega |Du| \leqslant \liminf_{k\to\infty} \int_\Omega |Du_k|.$$

Theorem 2.2. *Assume $\{u_k\}_{k=1}^\infty$ is a sequence in $BV(\Omega)$ satisfying $\sup_k \|u_k\|_{BV(\Omega)} < \infty$. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ and a function $u \in BV(\Omega)$ such that*

$$u_{k_j} \to u \quad \text{in } L^1(\Omega)$$

as $j \to \infty$.

Let us give the definition of weak solutions to problem (1.6)–(1.8) followed the idea of [14,3].

Definition 2.2. A function $u \in L^2([0,T]; BV(\Omega))$ is called a weak solution of (1.6)–(1.8) if $\partial_t u \in L^2(\Omega_T)$, $u(0) = u_0$ and u satisfies

$$\int_0^s \int_\Omega \partial_t u(v-u)\, dx\, dt + \int_0^s \int_\Omega |Dv| \geqslant \int_0^s \int_\Omega |Du| + \lambda \int_0^s \int_\Omega \big(fe^{-u} - 1\big)(v-u)\, dx\, dt \tag{2.1}$$

for all $v \in L^2([0,T]; BV(\Omega))$ and a.e. $s \in [0,T]$.

Remark 1. If (2.1) holds, by selecting $v = u + \gamma\phi$ for $\phi \in C_0^\infty(\Omega)$ and $\gamma \in \mathbb{R}$ we get that u is a solution to (1.6) in the sense of distributions.

3. The variational problem

In this section we discuss the following restoration model:

$$\inf_{u\in BV(\Omega)} \left\{ \int_\Omega |Du| + \lambda \int_\Omega \big(u + fe^{-u}\big) \right\}, \tag{3.1}$$

where $f \in L^\infty(\Omega)$ is the given data and $\lambda > 0$ is the weighted parameter. We prove that problem (3.1) has a unique solution in $BV(\Omega)$.

Theorem 3.1. *Let $f \in L^\infty(\Omega)$ with $\inf_\Omega f > 0$, then problem* (3.1) *admits a unique solution $u \in BV(\Omega)$ satisfying*

$$\inf_\Omega(\log f) \leqslant u \leqslant \sup_\Omega(\log f). \tag{3.2}$$

Proof. Denote by $\alpha = \inf_\Omega(\log f)$, $\beta = \sup_\Omega(\log f)$ and

$$E(u) = \int_\Omega |Du| + \lambda \int_\Omega \big(u + fe^{-u}\big).$$

Let us denote by

$$h(s) = s + fe^{-s}. \tag{3.3}$$

It is obvious that $h''(s) = fe^{-s}$ and h is strictly convex as $f > 0$. Therefore, we have

$$h(s) \geqslant 1 + \log f \geqslant 1 + \alpha$$

for all $s \in \mathbb{R}$. This implies that $E(u)$ has a lower bound for all $u \in BV(\Omega)$. Hence we consider a minimizing sequence $\{u_n\} \subset BV(\Omega)$ for (3.1).

First, we show that $\alpha \leqslant u_n \leqslant \beta$. Since $f \in L^\infty(\Omega)$ with $\inf_\Omega f > 0$, we can choose a sequence $\{f_n\} \subset C^\infty(\overline{\Omega})$ such that $f_n \to f$ in $L^1(\Omega)$ and a.e. in Ω as $n \to \infty$, and

$$\inf_\Omega f \leqslant f_n \leqslant \sup_\Omega f. \tag{3.4}$$

If in (3.3) f is replaced by f_n, we see that $h(s)$ is decreasing as $s \in (-\infty, \log f_n)$ and increasing as $s \in (\log f_n, +\infty)$ for $n \in N$. Therefore, if $M \geqslant \log f_n$, one always has

$$\min(s, M) + f_n e^{-\min(s,M)} \leqslant s + f_n e^{-s}$$

for $x \in \Omega$ and $n \in N$. Notice that

$$\beta = \sup_\Omega(\log f) \geqslant \log f_n$$

from (3.4). Hence, if we let $M = \beta$, we have

$$\int_\Omega \big(\inf(u, \beta) + f_n e^{-\inf(u,\beta)}\big) \leqslant \int_\Omega \big(u + f_n e^{-u}\big).$$

Letting $n \to \infty$ in the above inequality, using Lebesgue Convergence Theorem and (3.4), we conclude

$$\int_\Omega \big(\inf(u, \beta) + f e^{-\inf(u,\beta)}\big) \leqslant \int_\Omega \big(u + f e^{-u}\big). \tag{3.5}$$

Moreover, by using the results of [8] (see Lemma 1 in Section 4.3), we obtain

$$\int_\Omega \big|D\big(\inf(u, \beta)\big)\big| \leqslant \int_\Omega |Du|. \tag{3.6}$$

Combining (3.5) and (3.6), we deduce that

$$E\big(\inf(u, \beta)\big) \leqslant E(u).$$

On the other hand, in the same way we get that $E(\sup(u, \alpha)) \leqslant E(u)$. Therefore, we can assume that without restriction that $\alpha \leqslant u_n \leqslant \beta$.

Second, we prove that there exists $u \in BV(\Omega)$ such that

$$E(u) = \min_{u \in BV(\Omega)} E(u).$$

The above proof implies in particular that u_n is bounded in $L^1(\Omega)$. Moreover, by the definition of $\{u_n\}$, we get that there exists a constant C such that

$$\int_\Omega |Du_n| + \int_\Omega h(u_n) \leqslant C, \tag{3.7}$$

since $\alpha \leqslant u_n \leqslant \beta$ and $h \in C[\alpha, \beta]$, we get that $h(u_n)$ is bounded. Therefore, by using (3.7), we deduce that

$$\int_\Omega |Du_n| \leqslant C.$$

Hence, we get that u_n is bounded in $BV(\Omega)$ and there exists u in $BV(\Omega)$ such that up to a subsequence, $u_n \to u$ in $L^1(\Omega)$-strong. Necessarily, we have $\alpha \leqslant u \leqslant \beta$. By using the lower semicontinuity of the total variation and Fatou's Lemma, we get that u is a solution of problem (3.1).

Since h is strictly convex as $f > 0$, the uniqueness of the minimizer follows from the strict convexity of the energy functional in (3.1). $\square$

4. The associated evolution equation

In this section we study the weak solutions of the evolution equation (1.6) associated to (3.1). First we consider the following approximation **Problem $\mathbf{P}_{\mathbf{R}}^{\epsilon,\delta}$** of (1.6):

Problem $\mathbf{P}_{\mathbf{R}}^{\epsilon,\delta}$.

$$\begin{cases} \partial_t u = \epsilon \triangle u + \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{|\nabla u|^2+\epsilon^2}}\right) + \lambda\left(f e^{-[u]_R} - 1\right), & \text{on } \Omega_T, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \times [0,T], \\ u(0) = u_0^\delta, & \text{on } \Omega. \end{cases} \tag{4.1}$$

Here $[\cdot]_R$ is the truncated function defined as $[\eta]_R := \max\{-R, \min\{R,\eta\}\}$, R is a constant that will be determined in the latter part of this section; $u_0^\delta \in C^\infty(\overline{\Omega})$, $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$ such that $u_0^\delta \to u_0$ in $L^1(\Omega)$ and

$$\left\| u_0^\delta \right\|_{L^\infty(\Omega)} \leqslant C_1 \|u_0\|_{L^\infty(\Omega)}, \tag{4.2}$$

$$\int_\Omega \left|\nabla u_0^\delta\right| \leqslant C_2 \int_\Omega |Du_0|, \tag{4.3}$$

where C_1, C_2 is a fixed constant independent of δ.

We have the following existence and uniqueness results for **Problem $\mathbf{P}_{\mathbf{R}}^{\epsilon,\delta}$** (4.1).

Lemma 4.1. *Let $f \in L^\infty(\Omega)$. For fixed $\epsilon, \delta, R > 0$, approximation problem* (4.1) *admits a unique weak solution $u_R^{\epsilon,\delta}$ such that $u_R^{\epsilon,\delta} \in L^\infty(0,T;H^1(\Omega))$, $\partial_t u_R^{\epsilon,\delta} \in L^2(0,T;L^2(\Omega))$ and*

$$\begin{aligned} &\int_0^t \int_\Omega \left|\partial_t u_R^{\epsilon,\delta}\right|^2 dx\,dt + \epsilon \int_\Omega \left|\nabla u_R^{\epsilon,\delta}(t)\right|^2 dx + 2\int_\Omega \left|\nabla u_R^{\epsilon,\delta}(t)\right| dx \\ &\quad \leqslant \epsilon \int_\Omega \left|\nabla u_0^\delta\right|^2 dx + 2\int_\Omega \left|\nabla u_0^\delta\right| dx + C(R) + 2|\Omega|\epsilon \end{aligned} \tag{4.4}$$

for a.e. $t \in [0,T]$, where $C(R)$ is a constant dependent on R and $|\Omega|$ denotes the Lebesgue measure of Ω.

Proof. By using the Galerkin method and Lebesgue Convergence Theorem, the fact that $\frac{p}{\sqrt{p^2+\epsilon^2}}$ is a monotone operator [2] and $f e^{-[u]_R}$ is bounded, we get that problem (4.1) admits a unique weak solution $u_R^{\epsilon,\delta}$ such that

$$\partial_t u_R^{\epsilon,\delta} \in L^2\left(0,T;L^2(\Omega)\right), \qquad u_R^{\epsilon,\delta} \in L^\infty\left(0,T;H^1(\Omega)\right).$$

Moreover, multiplying the first equation in (4.1) by $\partial_t u_R^{\epsilon,\delta}$ and integrating it over Ω, we have

$$\begin{aligned} \int_\Omega \left|\partial_t u_R^{\epsilon,\delta}\right|^2 dx + \frac{d}{dt}\int_\Omega \left(\frac{\epsilon}{2}\left|\nabla u_R^{\epsilon,\delta}\right|^2 + \sqrt{\left|\nabla u_R^{\epsilon,\delta}\right|^2+\epsilon^2}\right) dx &= \lambda \int_\Omega \left(f e^{-[u_R^{\epsilon,\delta}]_R} - 1\right)\partial_t u_R^{\epsilon,\delta}\, dx \\ &\leqslant \frac{1}{2}\int_\Omega \left|\partial_t u_R^{\epsilon,\delta}\right|^2 dx + C(R). \end{aligned} \tag{4.5}$$

Since

$$|p| \leqslant \sqrt{|p|^2+\epsilon^2} \leqslant |p| + \epsilon$$

holds for $p \in \mathbb{R}^n$, we easily obtain (4.4) by using (4.5). $\square$

In order to get the existence of weak solution for the original problem (1.6)–(1.8), we need derive some uniform estimates of the solution $\{u_R^{\epsilon,\delta}\}$ for the approximation problem (4.1).

Lemma 4.2. *Assume that $f \in L^\infty(\Omega)$ with $\inf_\Omega f > 0$, $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$ and*

$$M := \|\log f\|_{L^\infty(\Omega)} + C_1 \|u_0\|_{L^\infty(\Omega)}. \tag{4.6}$$

Let $\{u_R^{\epsilon,\delta}\}$ be a weak solution for problem (4.1) *with $R \geqslant M$. Then we have*

$$\|u_R^{\epsilon,\delta}\|_{L^\infty(\Omega_T)} \leqslant M.$$

Here C_1 is the same constant in (4.2).

Proof. Since $u_R^{\epsilon,\delta}$ is a weak solution of (4.1), we see that $(u_R^{\epsilon,\delta} - M)_+(t) \in H^1(\Omega)$, where $(\cdot)_+$ is the truncated function defined as $(\zeta)_+ := \max\{0, \zeta\}$.

Multiplying the first equation in (4.1) by $(u_R^{\epsilon,\delta} - M)_+(t)$ and integrating it over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\left(u_R^{\epsilon,\delta}(t) - M\right)_+\right|^2 dx + \epsilon \int_\Omega \left|\nabla\left(u_R^{\epsilon,\delta} - M\right)_+\right|^2 + \int_\Omega \frac{|\nabla(u_R^{\epsilon,\delta} - M)_+|^2}{\sqrt{|\nabla u|^2 + \epsilon^2}} \\
&\quad \leqslant \lambda \int_\Omega \left(f e^{-[u_R^{\epsilon,\delta}]_R} - 1\right)\left(u_R^{\epsilon,\delta} - M\right)_+.
\end{aligned} \tag{4.7}$$

Note that if $R \geqslant M$, $u \geqslant M$, then

$$[u]_R \geqslant M \geqslant \|\log f\|_{L^\infty(\Omega)}.$$

Hence, we have

$$\begin{aligned}
\int_\Omega \left(f e^{-[u_R^{\epsilon,\delta}]_R} - 1\right)\left(u_R^{\epsilon,\delta} - M\right)_+ &= \int_{\{u_R^{\epsilon,\delta} \geqslant M\}} \left(f e^{-[u_R^{\epsilon,\delta}]_R} - 1\right)\left(u_R^{\epsilon,\delta} - M\right)_+ \\
&\leqslant \int_{\{u_R^{\epsilon,\delta} \geqslant M\}} \left(f e^{-\|\log f\|_{L^\infty}} - 1\right)\left(u_R^{\epsilon,\delta} - M\right)_+ \\
&\leqslant 0.
\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we conclude

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\left(u_R^{\epsilon,\delta}(t) - M\right)_+\right|^2 dx \leqslant 0.$$

The above inequality implies that

$$\int_\Omega \left|\left(u_R^{\epsilon,\delta}(t) - M\right)_+\right|^2 dx \leqslant 0$$

holds for a.e. $t \in [0.T]$, since $u_R^{\epsilon,\delta}(0) = u_0^\delta \leqslant M$ from (4.2) and (4.6). Therefore, we get that

$$u_R^{\epsilon,\delta}(x,t) \leqslant M$$

holds for a.e. $(x,t) \in \Omega_T$.

By similar argument, multiplying the first equation in (4.1) by $(-u_R^{\epsilon,\delta} - M)_+(t)$ we conclude

$$u_R^{\epsilon,\delta}(x,t) \geqslant -M$$

holds for a.e. $(x,t) \in \Omega_T$. Thus $\|u_R^{\epsilon,\delta}\|_{L^\infty(\Omega_T)} \leqslant M$. □

Remark 2. Choosing $R = M$ in (4.1), we see that the truncated function $[\cdot]_R$ in (4.1) can be omitted. In the following, we consider the solution of **Problem $\mathbf{P_R^{\epsilon,\delta}}$** as $u^{\epsilon,\delta}$ depending only on ϵ, δ (not on R). Moreover, choosing $R = M$ in (4.4), the solution $u^{\epsilon,\delta}$ has the following estimate

$$\int_0^t\int_\Omega \left|\partial_t u^{\epsilon,\delta}\right|^2 dx\,dt+\epsilon\int_\Omega \left|\nabla u^{\epsilon,\delta}(t)\right|^2 dx+2\int_\Omega \left|\nabla u^{\epsilon,\delta}(t)\right| dx$$
$$\leqslant \epsilon\int_\Omega \left|\nabla u_0^\delta\right|^2 dx+2\int_\Omega \left|\nabla u_0^\delta\right| dx+C+2|\Omega|\epsilon \tag{4.9}$$

for a.e. $t\in[0,T]$.

Now we are able to establish the existence and uniqueness theorem of weak solutions for problem (1.6)–(1.8).

Theorem 4.3. *Assume that $f\in L^\infty(\Omega)$ with $\inf_\Omega f>0$, and $u_0\in L^\infty(\Omega)\cap BV(\Omega)$. Then problem* (1.6)–(1.8) *admits a unique weak solution in the sense of Definition* 2.2.

Proof. Let $u^{\epsilon,\delta}$ be the solution to (4.1). By Lemma 4.6, Remark 2 and (4.9) we get that

$$\left\|u^{\epsilon,\delta}\right\|_{L^\infty(\Omega_T)}\leqslant M \tag{4.10}$$

and

$$\int_0^T\int_\Omega \left|\partial_t u^{\epsilon,\delta}\right|^2 dx\,dt+2\int_\Omega \left|\nabla u^{\epsilon,\delta}(t)\right| dx\leqslant \epsilon\int_\Omega \left|\nabla u_0^\delta\right|^2 dx+2\int_\Omega \left|\nabla u_0^\delta\right| dx+C+2|\Omega|\epsilon \tag{4.11}$$

for a.e. $t\in[0,T]$. It follows from (4.10) and (4.11) that

$$\left\|u^{\epsilon,\delta}\right\|_{BV(\Omega_T)}=\int_0^T\int_\Omega \left|u^{\epsilon,\delta}\right|+\int_0^T\int_\Omega\left(\left|\partial_t u^{\epsilon,\delta}\right|+\left|\nabla u^{\epsilon,\delta}\right|\right)\leqslant C\left\|u_0^\delta\right\|_{H^1(\Omega)},$$

where $C=C(\Omega,T)$. Therefore, combining the above inequality, (4.11) and Theorem 2.2, we conclude that for fixed $\delta>0$, there exists a subsequence $\{u^{\epsilon,\delta}\}$ such that

$$u^{\epsilon,\delta}\to u^\delta \quad \text{strongly in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T, \tag{4.12}$$
$$u^{\epsilon,\delta}\to u^\delta \quad \text{strongly in } L^1(\Omega) \text{ for a.e. } t\in[0,T], \tag{4.13}$$
$$\partial_t u^{\epsilon,\delta}\rightharpoonup \partial_t u^\delta \quad \text{weakly in } L^2(\Omega_T) \tag{4.14}$$

as $\epsilon\to0$. Moreover, letting $\epsilon\to0$ in (4.10) with fixed δ and using (4.12), we have

$$\left\|u^\delta\right\|_{L^\infty(\Omega_T)}\leqslant M. \tag{4.15}$$

Notice the fact that as $\epsilon\to0$,

$$u^{\epsilon,\delta}\to u^\delta \quad \text{strongly in } L^2(\Omega_T), \tag{4.16}$$

since it follows from (4.10) and (4.15) that

$$\int_0^T\int_\Omega \left|u^{\epsilon,\delta}-u^\delta\right|^2\leqslant C(M)\int_0^T\int_\Omega \left|u^{\epsilon,\delta}-u^\delta\right|.$$

Recall that

$$\begin{cases}\sqrt{p^2+\epsilon^2}-\sqrt{q^2+\epsilon^2}\geqslant \frac{q}{\sqrt{q^2+\epsilon^2}}\cdot(p-q),\\ p^2-q^2\geqslant 2q\cdot(p-q)\end{cases} \tag{4.17}$$

hold for $p,q\in\mathbb{R}^n$, due to the convexity of $\sqrt{p^2+\epsilon^2}$ and p^2. Since $u^{\epsilon,\delta}$ is a weak solution to (4.1), multiplying the first equation in (4.1) by $v-u^{\epsilon,\delta}$, using (4.17) and integrating it over $\Omega\times[0,s]$, we conclude that

$$
\begin{aligned}
&\int_0^s\int_\Omega \partial_t u^{\epsilon,\delta}\big(v-u^{\epsilon,\delta}\big)\,dx\,dt+\frac{\epsilon}{2}\int_0^s\int_\Omega |\nabla v|^2\,dx\,dt+\int_0^s\int_\Omega \sqrt{|\nabla v|^2+\epsilon^2}\,dx\,dt\\
&\quad\geqslant \frac{\epsilon}{2}\int_0^s\int_\Omega \big|\nabla u^{\epsilon,\delta}\big|^2\,dx\,dt+\int_0^s\int_\Omega \sqrt{\big|\nabla u^{\epsilon,\delta}\big|^2+\epsilon^2}\,dx\,dt\\
&\qquad+\lambda\int_0^s\int_\Omega \big(fe^{-u^{\epsilon,\delta}}-1\big)\big(v-u^{\epsilon,\delta}\big)\,dx\,dt\\
&\quad\geqslant \int_0^s\int_\Omega \big|\nabla u^{\epsilon,\delta}\big|\,dx\,dt+\lambda\int_0^s\int_\Omega \big(fe^{-u^{\epsilon,\delta}}-1\big)\big(v-u^{\epsilon,\delta}\big)\,dx\,dt
\end{aligned}
\tag{4.18}
$$

holds for all $v\in L^2(0,T;H^1(\Omega))$. Now considering every term in (4.18), and using (4.10), (4.12)–(4.14), (4.16) and Lebesgue Convergence Theorem, we obtain that

$$
\int_0^s\int_\Omega \partial_t u^{\epsilon,\delta}\big(v-u^{\epsilon,\delta}\big)\,dx\,dt\to\int_0^s\int_\Omega \partial_t u^{\delta}\big(v-u^{\delta}\big)\,dx\,dt,
\tag{4.19}
$$

$$
\frac{\epsilon}{2}\int_0^s\int_\Omega |\nabla v|^2\,dx\,dt+\int_0^s\int_\Omega \sqrt{|\nabla v|^2+\epsilon^2}\,dx\,dt\to\int_0^s\int_\Omega |\nabla v|\,dx\,dt
\tag{4.20}
$$

and

$$
\lambda\int_0^s\int_\Omega \big(fe^{-u^{\epsilon,\delta}}-1\big)\big(v-u^{\epsilon,\delta}\big)\,dx\,dt\to\lambda\int_0^s\int_\Omega \big(fe^{-u^{\delta}}-1\big)\big(v-u^{\delta}\big)\,dx\,dt
\tag{4.21}
$$

as $\epsilon\to 0$. Moreover, combining (4.13) and Theorem 2.1 we have

$$
\int_\Omega \big|Du^\delta(t)\big|\leqslant\liminf_{\epsilon\to 0}\int_\Omega \big|\nabla u^{\epsilon,\delta}(t)\big|\,dx\quad\text{for a.e. } t\in[0,T],
\tag{4.22}
$$

which implies

$$
\int_0^s\int_\Omega \big|Du^\delta\big|\,dt\leqslant\liminf_{\epsilon\to 0}\int_0^s\int_\Omega \big|\nabla u^{\epsilon,\delta}\big|\,dx\,dt
\tag{4.23}
$$

by using Fatou's Lemma. Therefore, combining (4.19)–(4.21) and (4.23), let $\epsilon\to 0$ in (4.18) to arrive at

$$
\begin{aligned}
&\int_0^s\int_\Omega \partial_t u^{\delta}\big(v-u^{\delta}\big)\,dx\,dt+\int_0^s\int_\Omega |\nabla v|\,dx\,dt\\
&\quad\geqslant\int_0^s\int_\Omega \big|Du^\delta\big|\,dt+\lambda\int_0^s\int_\Omega \big(fe^{-u^{\delta}}-1\big)\big(v-u^{\delta}\big)\,dx\,dt.
\end{aligned}
\tag{4.24}
$$

This shows that u^δ is a weak solution to (1.6) with initial data u_0^δ.

Additionally using (4.3), (4.22) and letting $\epsilon\to 0$ in (4.11), we obtain

$$
\int_0^T\int_\Omega \big|\partial_t u^\delta\big|^2\,dx\,dt+2\int_\Omega \big|Du^\delta(t)\big|\leqslant 2C_2\int_\Omega |\nabla u_0|
\tag{4.25}
$$

holds for a.e. $t\in[0,T]$. Similarly, we have

$$
\big\|u^\delta\big\|_{BV(\Omega_T)}+\big\|\partial_t u^\delta\big\|_{L^2(\Omega_T)}\leqslant C,
$$

where the constant C is independent of δ. Therefore, we can extract a subsequence in δ to get as $\delta \to 0$

$$u^\delta \to u \quad \text{in } L^1(\Omega_T), \text{ hence in } L^2(\Omega_T) \text{ from (4.15)},$$
$$u^\delta \to u \quad \text{in } L^1(\Omega), \text{ for a.e. } t \in [0,T],$$

and

$$\partial_t u^\delta \rightharpoonup \partial_t u \quad \text{in } L^2(\Omega_T). \tag{4.26}$$

Finally, by the similar argument pass to the limit as $\delta \to 0$ in (4.24) to get

$$\int_0^s \int_\Omega \partial_t u(v-u)\,dx\,dt + \int_0^s \int_\Omega |Dv| \geqslant \int_0^s \int_\Omega |Du| + \lambda \int_0^s \int_\Omega \big(fe^{-u}-1\big)(v-u)\,dx\,dt$$

for all $v \in L^2([0,T]; BV(\Omega))$ and a.e. $s \in [0,T]$. At the same time we see that $u \in L^\infty(0,T; BV(\Omega) \cap L^\infty(\Omega))$ and $\partial_t u \in L^2(\Omega_T)$ from (4.15), (4.25) and (4.26). Thus we get the existence of a weak solution u to problem (1.6)–(1.8).

In the following we prove the uniqueness of weak solutions to problem (1.6)–(1.8). Let u_1, u_2 be two weak solutions to (1.6) with $u_1(0) = u_2(0) = u_0$. Then we have

$$\int_0^s \int_\Omega \partial_t u_1(u_2-u_1)\,dx\,dt + \int_0^s \int_\Omega |Du_2| \geqslant \int_0^s \int_\Omega |Du_1| + \lambda \int_0^s \int_\Omega \big(fe^{-u_1}-1\big)(u_2-u_1)\,dx\,dt$$

and

$$\int_0^s \int_\Omega \partial_t u_2(u_1-u_2)\,dx\,dt + \int_0^s \int_\Omega |Du_1| \geqslant \int_0^s \int_\Omega |Du_2| + \lambda \int_0^s \int_\Omega \big(fe^{-u_2}-1\big)(u_1-u_2)\,dx\,dt.$$

Adding the above two inequalities we get

$$\int_0^s \int_\Omega \partial_t(u_1-u_2)(u_1-u_2)\,dx\,dt \leqslant -\lambda \int_0^s \int_\Omega f\big(e^{-u_1}-e^{-u_2}\big)(u_1-u_2)\,dx\,dt.$$

Since $\lambda > 0$, $f > 0$ and e^s is a monotone function for $s \in \mathbb{R}$, it follows from the above inequality that

$$\frac{1}{2}\int_0^s \frac{d}{dt}\int_\Omega |u_1-u_2|^2\,dx\,dt \leqslant 0.$$

This implies

$$\big\|u_1(\cdot,s) - u_2(\cdot,s)\big\|_{L^2(\Omega)} = 0$$

for a.e. $s \in [0,T]$. Therefore $u_1 = u_2$. □

5. Numerical results

In this section some numerical tests on our model (1.5) are demonstrated for multiplicative noise removal to compare with the known model (AA). Numerically we get a solution to problem (3.1) by computing the associated Eq. (1.6) to a steady state. To discretize equation (1.6), the finite difference scheme in [12] is used. Denote the space step by $h = 1$ and the time step by τ. Thus we have

$$D_x^{\pm}(u_{i,j}) = \pm[u_{i\pm1,j} - u_{i,j}],$$
$$D_y^{\pm}(u_{i,j}) = \pm[u_{i,j\pm1} - u_{i,j}],$$
$$\big|D_x(u_{i,j})\big| = \sqrt{\big(D_x^+(u_{i,j})\big)^2 + \big(m\big[D_y^+(u_{i,j}), D_y^-(u_{i,j})\big]\big)^2 + \delta},$$
$$\big|D_y(u_{i,j})\big| = \sqrt{\big(D_y^+(u_{i,j})\big)^2 + \big(m\big[D_x^+(u_{i,j}), D_x^-(u_{i,j})\big]\big)^2 + \delta},$$

where $m[a,b] = (\frac{\text{sign}a+\text{sign}b}{2}) \cdot \min(|a|,|b|)$ and $\delta > 0$ is the regularized parameter chosen near 0.

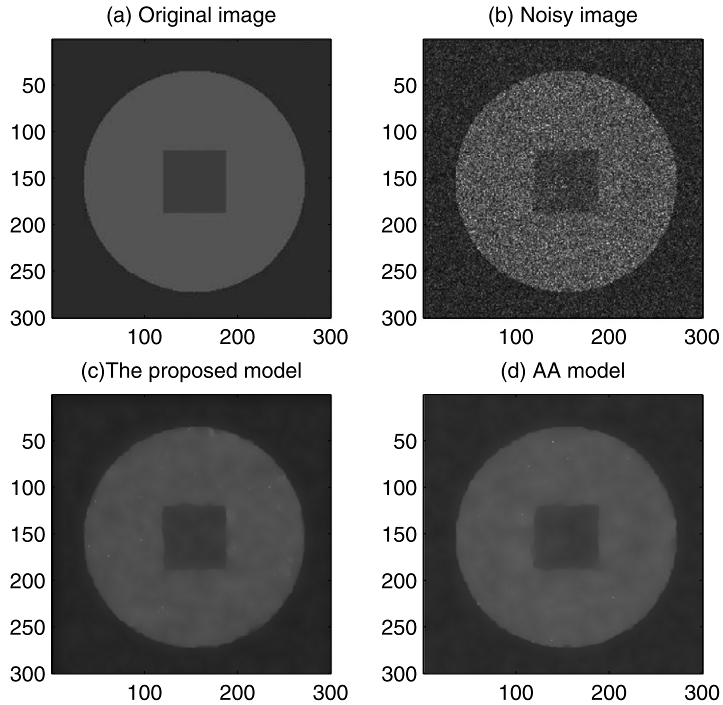

Fig. 1. (b) The noisy image is corrupted by some multiplicative noise with Gamma law of mean one, SNR = 3.09; (c) SNR = 26.76, $\lambda = 0.5$, iterations = 250; (d) SNR = 22.46, $\lambda = 2$, iterations = 260.

The numerical algorithms for Eq. (1.6) are given in the following (the subscripts i, j are omitted):

$$\frac{u^{n+1}-u^n}{\tau} = \left[D_x^-\left(\frac{D_x^+u^n}{|D_xu^n|}\right) + D_y^-\left(\frac{D_y^+u^n}{|D_yu^n|}\right)\right] + \lambda\left(fe^{-u^n}-1\right),$$

with boundary conditions

$$u_{0,j}^n = u_{1,j}^n, \qquad u_{N,j}^n = u_{N-1,j}^n, \qquad u_{i,0}^n = u_{i,1}^n, \qquad u_{i,N}^n = u_{i,N-1}^n$$

for $i, j = 1, \dots, N-1$.

The parameters are chosen like this: $\tau = 0.2$, $\delta = 0.0001$, and the larger the noise is, the smaller the fidelity coefficient λ is. In addition, we take $u^0 = f$ as the initial value. Similarly, we carry out numerical experiments for AA model (1.2) by discretizing the corresponding evolution (1.3) with the above numerical algorithm. In the following numerical experiments, for each scheme it is will be stopped at the index where the variance of the recovered noise matches that of our prior knowledge.

In order to evaluate the two models, we show the Signal to Noise Ratio (SNR) of the restored image. For a given true image u and its noisy observation u_0, the noise is denote by $n = u_0 - u$. With this we define the SNR in dB as

$$\mathrm{SNR} = 20\log_{10}\left(\frac{\int_\Omega (u_0-\bar{u}_0)^2\,dx\,dy}{\int_\Omega (n-\bar{n})^2\,dx\,dy}\right),$$

where

$$\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0\,dx\,dy, \qquad \bar{n} = \frac{1}{|\Omega|}\int_\Omega n\,dx\,dy.$$

In Figs. 1 and 2, two synthetic images are corrupted by some multiplicative noise with Gamma law of mean 1. We display the denoising results obtained by our approach, as well as with AA model. We see that our model gets a good visual effect, and has a higher SNR than AA model.

In Figs. 3 and 4, two real images are corrupted by some multiplicative Gaussian noise with the standard derivation $\sigma = 0.2$, and the original image in Fig. 4 includes plenty of textures. We see that our model also has a higher SNR than AA model, and works well for the texture image.

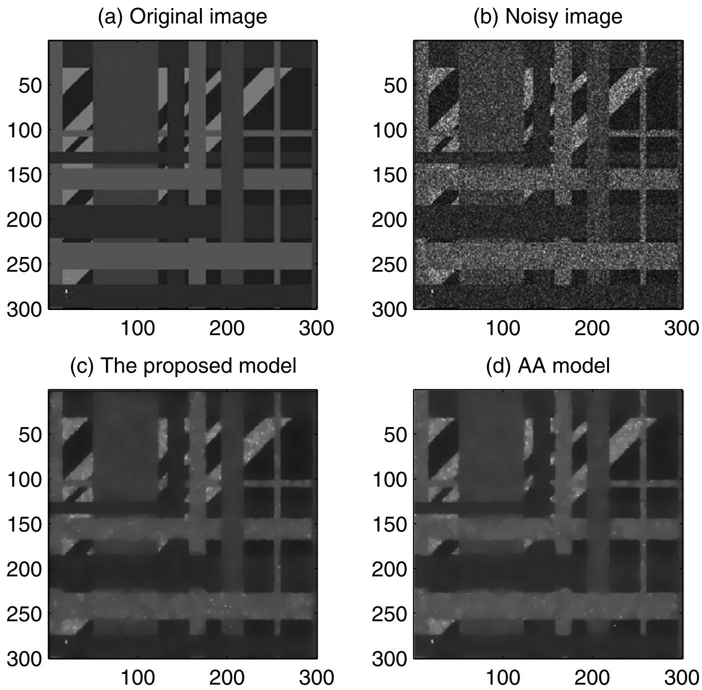

Fig. 2. (b) The noisy image is corrupted by some multiplicative noise with Gamma law of mean one, SNR $= 3.98$; (c) SNR $= 15.34$, $\lambda = 0.5$, iterations $= 180$; (d) SNR $= 12.64$, $\lambda = 13$, iterations $= 180$.

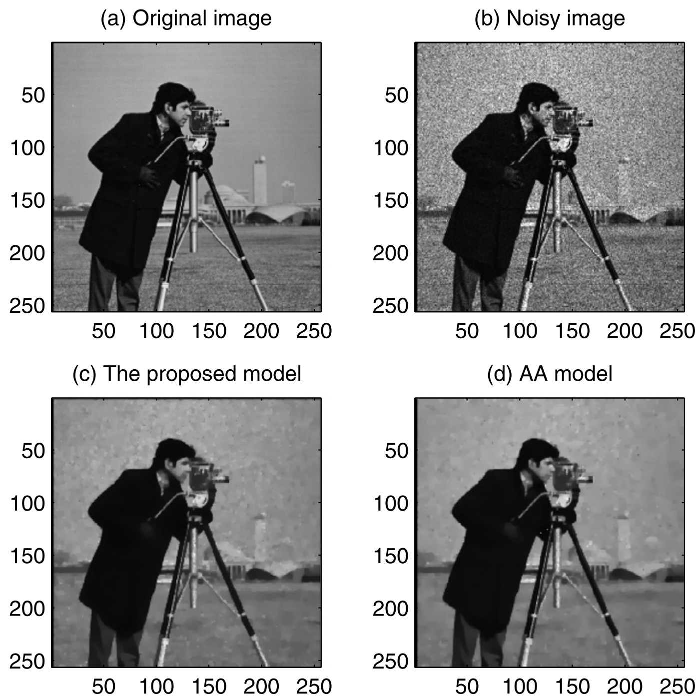

Fig. 3. (b) The noisy image is corrupted by some multiplicative Gaussian noise with $\sigma = 0.2$, SNR $= 16.38$; (c) SNR $= 28.78$, $\lambda = 0.1$, iterations $= 80$; (d) SNR $= 23.93$, $\lambda = 190$, iterations $= 100$.

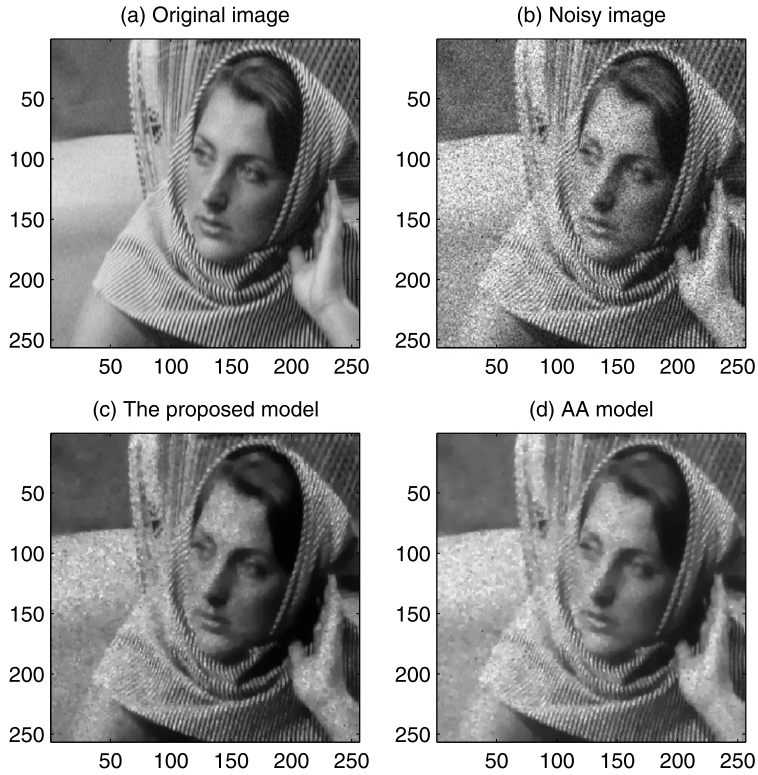

Fig. 4. (b) The noisy image is corrupted by some multiplicative Gaussian noise with $\sigma = 0.2$, SNR $= 14.47$; (c) SNR $= 24.11$, $\lambda = 0.8$, iterations $= 70$; (d) SNR $= 20.90$, $\lambda = 500$, iterations $= 80$.

Acknowledgment

The first author is very grateful to Professor Michael K.P. Ng for fruitful discussions.

References

[1] G. Aubert, J.F. Aujol, A variational approach to removing multiplicative noise, SIAM J. Appl. Math. 68 (2008) 925–946.
[2] H. Brezis, Operateures Maximaux Monotone, North-Holland, Amsterdam, 1993.
[3] Y.M. Chen, T. Wunderli, Adaptive total variation for image restoration in BV space, J. Math. Anal. Appl. 272 (2002) 117–137.
[4] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.
[5] F. Giusti, Minimal Surfaces and Functions of Bounded Variation, Monogr. Math., vol. 80, Brikhäuser, Basel, 1984.
[6] Y. Huang, M. Ng, Y. Wen, A new total variation method for multiplicative noise removal, SIAM J. Imaging Sci. 2 (1) (2009) 20–40.
[7] Y. Huang, M. Ng, Y. Wen, A fast total variation minimization method for image restoration, SIAM J. Multiscale Model. Simul. 7 (2008) 774–795.
[8] P. Kornprobst, R. Deriche, G. Aubert, Image sequence analysis via partial differential equations, J. Math. Imaging Vision 11 (1) (1999) 5–26.
[9] F.H. Lin, X.P. Yang, Geometric Measure Theory – An Introduction, Science Press/International Press, Beijing/Boston, 2002.
[10] M. Lysaker, A. Lundervold, X.C. Tai, Noise removal using fourth-order partial differential equation with application to medical magnetic resonance images in space and time, IEEE Trans. Image Process. 12 (12) (2003) 1579–1590.
[11] L. Rudin, P.-L. Lions, S. Osher, Multiplicative denoising and deblurring: Theory and algorithms, in: S. Osher, N. Paragios (Eds.), Geometric Level Sets in Imaging, Vision, and Graphics, Springer, 2003, pp. 103–119.
[12] L. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, Phys. D 60 (1992) 259–268.
[13] M. Tur, C. Chin, J.W. Goodman, When is speckle noise multiplicative? Appl. Optics 21 (7) (1982) 1157–1159.
[14] X. Zhou, An evolution problem for plastic antiplanar shear, Appl. Math. Optim. 25 (1992) 263–285.